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SINCLAIR TYPE INEQUALITIES FOR THE LOCAL SPECTRAL RADIUS AND RELATED TOPICS

BY

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ABSTRACT

We generalize the well-known Sinclair lemma for Hermitian elements, proving pointwise versions for generalized scalar operators and unbounded skew-Hermitian operators.

0. Introduction

Using Bernstein's inequality (see [4, Theorem 11.1.2]):

$$\sup_{t\in R} |F'(t)| \leqq \tau \sup_{t\in R} |F(t)| \tag{0.1}$$

for entire functions $F(z)$ of exponential type τ bounded on R, Browder [7], Sinclair [21] and Katznelson [15] proved the important Sinclair lemma: $\|A\| = r(A)$ for Hermitian operators A on a Banach space X ($r(\cdot)$ — the spectral radius; see also [5, §26]).

We develop this idea further, to what seems to be its natural extent, and prove here stronger results for a wider class of operators.

In Theorem 1, we apply an extension of Bernstein's inequality to obtain a general pointwise inequality for operators A such that

$$\|e^{itA}\| = O(|t|^p) \qquad \text{for some } p \geqq 0 \quad (t \in R, |t| \to \infty).$$

In particular:

$$\|Ax\| \leqq \|x\| \, r(A + iB, x) \tag{0.2}$$

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holds for any pair A, B of commuting Hermitian operators on X and every $x \in X$, where

$$(0.3) \qquad r(T,x) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$$

is the local spectral radius of T at x.

In Section 2 we prove the pointwise inequalities

$$(0.4) \qquad \|Ax\| \leqq \|x\| \liminf_{n\to\infty} \|A^n x\|^{1/n}, \qquad x \text{ analytic for } A,$$

for any unbounded closed skew-Hermitian operator A and

$$(0.5) \qquad \|Ax\| \leqq \frac{e}{2}\|x\| \liminf_{n\to\infty} (n\|A^n x\|^{1/n}), \qquad x \text{ geometric for } A$$

for any closed dissipative operator A. These inequalities follow from the operator versions (Theorem 2) of Landau–Kolmogorov's inequalities for C^∞ functions on R and R^+.

At the end of the paper we show how $r(T,x)$ defined in (0.3) can be used to construct hyperinvariant subspaces for a class of operators.

1. The local spectral radius and inequalities for generalized scalar operators

Throughout X denotes a complex Banach space and $B(X)$ the space of bounded linear operators on X. The quantity (0.3) defined for every $x \in X$ and $T \in B(X)$ is finite, as

$$\|T^n x\|^{1/n} \leqq \|T^n\|^{1/n} \|x\|^{1/n} \qquad (n = 1,2,\dots)$$

and hence $r(T,x) \leqq r(T) < \infty$. For every $\alpha, \beta \in \mathbf{C}$, $x, y \in X$ it easily follows that

$$(1.1) \qquad r(\alpha T, \beta x) = |\alpha|\, r(T,x)$$

and

$$(1.2) \qquad r(T, x+y) \leqq \max\{r(T,x), r(T,y)\}.$$

An operator $T \in B(X)$ is said to have the single-valued extension property (SVEP) if whenever $(zI - T)g(z) = 0$ holds for some X-valued holomorphic function g in some open set $U \subseteq C$, then $g = 0$ in U. If T has the SVEP, for every $x \in X$ the holomorphic function $(zI - T)^{-1}x$ has a single-valued maximal extension from the resolvent set of T to the open local resolvent set $\rho_T(x)$. Its complement $\sigma_T(x) = C \setminus \rho_T(x)$ is called the local spectrum of T at x. One easily sees that

$$r(T,x)=\sup\{|z|: z\in\sigma_T(x)\}$$

for every $x\in X$ when T has the SVEP (as $r(T,x)$ is the radius of convergence of $g_x(z)=\sum_{n=0}^{\infty}T^n x/z^{n+1}$ and $(zI-T)g_x(z)=x$ when $|z|>r(T,x)$).

Note that the generalized scalar operators considered later in this section and the hyponormal operators mentioned in Section 3 all have the SVEP ([10, p. 62], [9, p. 13]).

Let $g(z)$ be an entire function. If one of the quantities

$$\limsup_{|z|\to\infty}(\log|g(z)|)/|z|;\quad \limsup_{n\to\infty}|g^{(n)}(0)|^{1/n}$$

is finite, the other is finite too and they coincide. This number, say τ, is called the exponential type of g — see [4, Ch. 2], [17, Ch. 1]; τ is characterized also by the property: for every $\varepsilon>0$ there exists $M(\varepsilon)>0$ such that $|g(z)|\leqq M(\varepsilon)e^{(\tau+\varepsilon)|z|}$ when $z\in\mathbf{C}$. Note that $\tau\geqq 0$. The same is true for entire functions $g:\mathbf{C}\to X$, with $|\cdot|$ replaced by $\|\cdot\|$.

Definition. An entire function $g(z)$ belongs to the class P, if g is of finite exponential type, has no zeros in $\{\mathrm{Im}(z)<0\}$ and

$$\limsup_{t\to\infty}(\log|g(it)|)/t\leqq\limsup_{t\to\infty}(\log|g(-it)|)/t.$$

The next result was proved by Levin [17, IX.11], [4, 11.7.2].

Theorem. *Let g be in the class P and of exponential type τ. If $|F(t)|\leqq|g(t)|$ on R for some entire function $F(z)$ of exponential type $\sigma\leqq\tau$, then $|F^{(n)}(t)|\leqq|g^{(n)}(t)|$ on R, $n=1,2,\ldots$.*

We shall apply this theorem to obtain an inequality for the so-called generalized scalar (G-S) operators. For all necessary information about them we refer to [10]. One has the characterization: $A\in B(X)$ is G-S with real spectrum if and only if there exists $p\geqq 0$ such that $\|e^{itA}\|=O(|t|^p)$ when $t\in R$, $|t|\to\infty$ ([10, 5.4.5]).

Definitions. For every G-S operator A with real spectrum we denote by $p=p(A)$ the smallest integer with the above property and call it the degree of A. As $\|e^{itA}\|/|t-i|^{p(A)}$ is bounded on R, let $M=M(A)$ be the smallest constant in $\|e^{itA}\|\leqq M|t-i|^p$ $(t\in R)$.

Now we prove one of the main results.

THEOREM 1. *Let A, B be two G-S operators with real spectra. Then for every $x \in X$ and $n = 1, 2, \ldots$ one has*

$$\|A^n x\| \leqq M(A)\|x\| K(n, p(A), r) \tag{1.3}$$

where $r = r(A + iB, x) + c(A, B)$ and

$$K(n, p, r) = \sum_{k=0}^{n} \binom{n}{k} p(p-1)\cdots(p-n+k+1) r^k,$$

$$c(A, B) = \limsup_{t \in R, |t| \to \infty} (\log \| e^{tA} e^{-tA - itB} \|)/|t|.$$

The number $c(A, B)$ is finite and nonnegative (see [4, 5.4.4]); *when A and B commute, $c(A, B) = 0$.*

PROOF. Let $x \in X$ and $h \in X'$, $\|h\| = 1$. We consider the functions:

$$F(z) = h(e^{izA}x) \quad \text{and} \quad g(z) = M(A)\|x\|(z - i)^{p(A)} e^{izr}, \qquad z \in \mathbf{C}.$$

It is clear that g is in the class P and of exponential type r.

For $t \in R$: $|F(t)| \leqq \|e^{itA}x\| \leqq M(A)\|x\| \, |t - i|^{p(A)} = |g(t)|$. Also

$$\limsup_{|t| \to \infty} (\log \|e^{itA}\|)/|t| = 0$$

and

$$\begin{aligned} \limsup_{|t| \to \infty} (\log \|e^{tA}x\|)/|t| &\leqq \limsup_{|t| \to \infty} (\log \|e^{tA} e^{-tA - itB}\|)/|t| \\ &\quad + \limsup_{|t| \to \infty} (\log \|e^{t(A+iB)}x\|)/|t| \\ &\leqq c(A, B) + r(A + iB, x) \\ &= r. \end{aligned}$$

Let now $z = |z| e^{i\theta} \in \mathbf{C}$ be arbitrary. We have

$$\begin{aligned} \limsup_{|z| \to \infty} (\log \|e^{izA}x\|)/|z| &\leqq \limsup_{|z| \to \infty} (\log \|e^{i|z|\cos\theta A}\|)/|z| \\ &\quad + \limsup_{|z| \to \infty} (\log \|e^{-|z|\sin\theta A}x\|)/|z| \\ &\leqq |-\sin\theta| r \\ &\leqq r. \end{aligned}$$

As $|F(z)| \leqq \| e^{izA}x \|$, the exponential type of $F(z)$ does not exceed r. Applying the theorem of Levin to $F(z)$ and $g(z)$ we obtain:

$$|F^{(n)}(0)| = |h(A^n x)| \leqq |g^{(n)}(0)| = M(A)\|x\| K(n, p(A), r), \qquad n = 1, 2, \ldots,$$

and as h is arbitrary, the proof is completed.

COROLLARY 1. *If A, B are commuting G-S operators with real spectra, then for every $x \leqq X$ and $n = 1, 2, \ldots$*

$$\|A^n x\| \leqq M(A)\|x\| K(n, p(A), r), \quad \|B^n x\| \leqq M(B)\|x\| K(n, p(B), r) \tag{1.4}$$

with $K(n, p, r)$ as in Theorem 1 and $r = r(A + iB, x)$.

PROOF. We have $r(B - iA, x) = r(A + iB, x)$ according to (1.1).

COROLLARY 2. *Let A, B be two commuting G-S operators with real spectra of degrees p, q respectively and let $T = A + iB$. If $r(T, x) = 0$ for some $x \in X$, then $A^{p+1}x = B^{q+1}x = T^{p+q+1}x = 0$.*

PROOF. When $r = 0$, $K(p+1, p, 0) = K(q+1, q, 0) = 0$ and (1.4) implies $A^{p+1}x = B^{q+1}x = 0$.

For every $z \in \mathbf{C}$:

$$e^{zT}x = e^{zA}e^{izB} = e^{zA}\left(\sum_{k=0}^{q} (iz)^k B^k x / k!\right) = \left(\sum_{k=0}^{q} (iz)^k B^k / k!\right) e^{zA}x$$

$$= \left(\sum_{k=0}^{q} (iz)^k B^k / k!\right)\left(\sum_{m=0}^{p} z^m A^m x / m!\right).$$

Therefore $e^{zT}x$ is a X-valued polynomial in z of degree not exceeding $p + q$. This implies $T^{p+q+1}x = 0$.

COROLLARY 3 ([10, 4.3.5]). *If Q is a generalized scalar quasinilpotent operator, then Q is nilpotent.*

Note that every G-S operator $T \in B(x)$ has a decomposition $T = A + iB$ with A, B commuting G-S operators with real spectra ([10, 4.6.1]). Conversely, any such decomposition determines a G-S operator (see [10, 4.3.4]).

Let T_1, T_2 be two G-S operators. Define the operator $C(T_1, T_2)$ on $B(X)$ as follows: $C(T_1, T_2)(S) = T_1 S - S T_2$ ($S \in B(X)$). One of the important results in [10] says that if $\|C^n(T_1, T_2)(S)\|^{1/n} \to 0$ ($n \to \infty$) for some $S \in B(X)$, then $C^k(T_1, T_2)(S) = 0$ for some integer k (Theorem 4.4.5). We can now specify this integer. Let $T_1 = A + iB$, $T_2 = C + iD$ with A, B, C, D all G-S operators with real spectra and $AB = BA$, $CD = DC$. We have $C(T_1, T_2) =$

$C(A,C)+iC(B,D)$ and $C(A,C)$, $C(B,D)$ commute. It is easy to see that

$$e^{itC(A,C)}(S)=e^{itA}Se^{-itC} \qquad (t\in R,\ S\in B(X))$$

and hence $\|e^{itC(A,C)}\|\leqq\|e^{itA}\|\,\|e^{-itC}\|$. Therefore $C(A,C)$ is a G-S operator with real spectrum and $p(C(A,C))\leqq p(A)+p(C)$. Similarly $p(C(B,D))\leqq p(B)+p(D)$. If now $\|C^n(T_1,T_2)(S)\|^{1/n}\to 0$ $(n\to\infty)$ for some $S\in B(X)$, then Corollary 2 implies

$$C^{p+1}(A,C)(S)=C^{q+1}(B,D)(S)=C^k(T_1,T_2)(S)=0$$

where $p=p(A)+p(C)$, $q=p(B)+p(D)$, $k=p+q+1$.

In one particular case the inequality (1.3) has a very simple form — when A, B are Hermitian equivalent operators and $n=1$.

DEFINITIONS. A G-S operator A with real spectrum for which $p(A)=0$ is called Hermitian equivalent, and if in addition $M(A)=1$, i.e. $\|e^{itA}\|=1$ $(t\in R)$, A is called Hermitian ([5]). $T\in B(X)$ is called normal (normal equivalent), if $T=A+iB$ with A, B commuting Hermitian (Hermitian equivalent) operators on X.

COROLLARY 4. *If $T=A+iB$ is normal equivalent, then for all $x\in X$*

$$\|Ax\|\leqq M(A)\|x\|\,r(T,x),\quad \|Bx\|\leqq M(B)\|x\|\,r(T,x). \tag{1.5}$$

As $r(T,x)\leqq r(T)$, we obtain $\|A\|\leqq M(A)r(A+iB)$. When $B=0$ and $M(A)=1$, this is the Sinclair lemma.

COROLLARY 5 (Albrecht [1]). *If $T=A+iB$ is normal equivalent and $\|T^nx\|^{1/n}\to 0$ $(n\to\infty)$ for some $x\in X$, then $Ax=Bx=0$.*

COROLLARY 6. *Let T be a bounded linear operator on the Hilbert space H with polar decomposition $T=U|T|$. Then for every $x\in H$*

$$\|Tx\|\leqq\|x\|\,r(|T|,x). \tag{1.6}$$

2. Inequalities for unbounded skew-Hermitian and dissipative operators

Let A be a linear operator with domain $D(A)\subseteq X$ and let

$$D^\infty(A)=\{x\subseteq D(A)\colon A^nx\in D(A) \text{ for } n=1,2,\ldots\}.$$

DEFINITIONS. The operator A is said to be dissipative, if

$$\|tx-Ax\|\geqq t\|x\| \qquad (t\in R^+, x\in D(A)) \tag{2.1}$$

and skew-Hermitian, if A and $-A$ are dissipative. (When $A \in B(X)$, A is skew-Hermitian if and only if iA is Hermitian — see [5].)

An element $x \in D^{\infty}(A)$ is called analytic for A if the function $\sum_{n=0}^{\infty} z^n \|A^n x\|/n!$ is holomorphic in some disk $\{|z| < t_x\}$, $t_x > 0$. The set of analytic elements for A is denoted by $a(A)$.

Also, $x \in D^{\infty}(A)$ is called a geometric element for A, if

$$r(A, x) = \limsup_{n \to \infty} \|A^n x\|^{1/n} < \infty \tag{2.2}$$

in which case the function

$$F(A, z)x = \sum_{n=0}^{\infty} z^n A^n x/n! \qquad (z \in \mathbf{C}) \tag{2.3}$$

is entire and of exponential type $r(A, x)$. The set of geometric elements for A is denoted by $b(A)$.

When $x, y \in b(A)$, the properties (1.1) and (1.2) hold, hence the sets $b(A)$ and $K(A, a) = \{x \in b(A): r(A, x) \leqq a\}$ for every $a > 0$ are linear subspaces of $D^{\infty}(A)$, invariant for A.

We are interested in the case when $F(A, t)x$ is bounded on R or on R^+. Sufficient conditions are given by the following two lemmas.

Lemma 1. *If A is closed and skew-Hermitian, then for every $x \in a(A)$ the function $F(A, t)x$, $t \in (-t_x, t_x)$ can be extended to R as a C^{∞} function (denoted again by $F(A, t)x$) such that for $t \in R$, $\|F(A, t)x\| = \|x\|$ and*

$$(d^n/dt^n)F(A, t)x = F(A, t)A^n x, \qquad n = 1, 2, \ldots . \tag{2.4}$$

Moreover, $a(A)$ is invariant for A^n and $F(A, t)$.

The proof is contained in [6, Theorem 2].

Lemma 2. *Let A be a closed dissipative operator and $x \in b(A)$. Then $\|F(A, t)x\| \leqq \|x\|$ for $t \in R^+$, $b(A)$ is invariant for A^n and $F(A, t)$ ($n = 1, 2, \ldots$, $t \in R^+$) and (2.4) holds.*

Proof. Consider the operator-valued functions

$$G(z) = \sum_{n=0}^{\infty} A^n/z^{n+1}, \quad G_n(z) = \sum_{m=0}^{n} A^m/z^{m+1}.$$

When $|z| > r(A, x)$, $G(z)x$ is holomorphic and for every $k = 1, 2, \ldots$

$$G_n(z)A^k x = A^k G_n(z)x = z^k(G_{n+k}(z)x - G_{k-1}(z)x).$$

As A^k is closed ([12, 7.9.7]) and $G_n(z)y \to G(z)y$ ($y \in b(A)$, $n \to \infty$) we obtain $G(z)x \in D(A^k)$ and $z^k G_{k-1}(z)x = (z^k I - A^k)G(z)x = G(z)(z^k I - A^k)x$, so that $G(z)x \in D^\infty(A)$ and $x = (zI - A)G(z)x = G(z)(zI - A)x$ ($k = 1$).

According to (2.1), we have for every real $t > r(A, x)$:

$$\| tG(t)x \| \leqq \| x \| \tag{2.5}$$

and as $\| A^n G(t)x \|^{1/n} = \| G(t)A^n x \|^{1/n} \leqq t^{-1/n} \| A^n x \|^{1/n}$ ($n = 1, 2, \ldots$) we find $r(A, G(t)x) \leqq r(A, x)$.

Let now $a > 0$ be arbitrary and $t > a$, $x \in K(A, a)$.

First, it is clear that $G(t)$ maps $K(A, a)$ in itself. Also,

$$G(t)x - G(s)x = (s - t)G(t)G(s)x \qquad \text{when } s > a,$$

hence

$$(d^n/dt^n)G(t)x = (-1)^n n!\, G^{n+1}(t)x, \qquad n = 1, 2, \ldots .$$

As $F(A, z)x$ is of exponential type $r(A, x) \leqq a$, for every $\varepsilon > 0$ we have the estimate $\| F(A, s)x \| \leqq M(\varepsilon) e^{(a+\varepsilon)|s|}$, $s \in R$. Hence the integral $\int_0^\infty e^{-ts} F(A, s)x\,ds$ converges and integration by parts shows that it equals $G(t)x$.

We can apply now the Post–Widder inversion formula [23, Ch. 7] (which holds for X-valued functions via Hahn–Banach's theorem):

$$F(A, t)x = \lim_{n \to \infty} (-1)^n (n!)^{-1} (n/t)^{n+1} G^{(n)}(n/t)x$$

$$= \lim_{n \to \infty} (n/t)^{n+1} G^{n+1}(n/t)x,$$

and in view of (2.5), $\| F(A, t)x \| \leqq \| x \|$ ($t > a > 0$). As a is arbitrary, this holds for every $t \in R^+$. It easily follows that $F(A, t)$ keeps $K(A, a)$ invariant when $a > 0$, $t \in R^+$, therefore $F(A, t)$ keeps invariant $b(A)$ and (2.4) holds.

Remark. When A is densely defined, instead of $x \in b(A)$ we can assume only that x is an entire element for A, i.e. $\| A^n x \|^{1/n} = o(n)$ ($n \to \infty$), according to [18, Theorem 3.2], [3] and [12, 5.9.5].

We shall now apply to $F(A, t)x$ the Landau–Kolmogorov inequalities for functions on the real line R and on the half-line R^+:

$$\| f^{(k)} \| \leqq C_{n,k} \| f \|^{1-k/n} \| f^{(n)} \|^{k/n}, \tag{2.6}$$

$$\| f^{(k)} \| \leqq C^+_{n,k} \| f \|^{1-k/n} \| f^{(n)} \|^{k/n}, \qquad n = 2, 3, \ldots, \quad 1 \leqq k < n. \tag{2.7}$$

Here (2.6), (2.7) hold for any $f \in C^\infty(R)$ (resp. $f \in C^\infty(R^+)$) bounded there together with its derivatives and $\|\cdot\|$ is the "sup" norm. The best constants $C_{n,k}$ in (2.6) were found by Kolmogorov in explicit form; in particular, $C_{n,1} \to 1$ $(n \to \infty)$ — see [16], [19]. The best constants $C^+_{n,k}$ in (2.7) are not known in explicit form, but can be computed with any prescribed error ([20]).

Operator versions of these inequalities follow easily.

THEOREM 2. *Let A be a closed operator on X. Then*

$$\| A^k x \| \leqq C_{n,k} \| x \|^{1-k/n} \| A^n x \|^{k/n} \tag{2.8}$$

when A is skew-Hermitian and x is analytic for it; also

$$\| A^k x \| \leqq C^+_{n,k} \| x \|^{1-k/n} \| A^n x \|^{k/n}, \qquad n = 2, 3, \ldots, \quad 1 \leqq k < n, \tag{2.9}$$

when A is dissipative and x is geometric for A.

PROOF. For x given, we apply (2.6) and (2.7) to $f(t) = h(F(A, t)x)$, where $h \in X'$, $\| h \| = 1$ is arbitrary and $t \in R$ (resp. $t \in R^+$). As

$$| f^{(k)}(t) | = | h(F(A, t) A^k x) | \leqq \| A^k x \|, \quad f^{(k)}(0) = h(A^k x), \qquad k = 1, 2, \ldots,$$

(see Lemmas 1 and 2) the inequalities (2.8), (2.9) follow.

COROLLARY 7. *When A is closed, skew-Hermitian and $x \in a(A)$:*

$$\| Ax \| \leqq \| x \| \liminf_{n \to \infty} \| A^n x \|^{1/n}. \tag{2.10}$$

PROOF. Put $k = 1$ and $n \to \infty$ in (2.8). Cf. [19, Remark 6].

COROLLARY 8 (A boundedness criterion). *Let A be a closed skew-Hermitian operator. Then A is bounded if and only if $r(A, x) \leqq C$ on a subset of $b(A)$ which is dense in X.*

PROOF. When $A \in B(X)$, clearly $r(A, x) \leqq r(A)$ for every $x \in X$. Conversely, the assumption implies that the subspace $K(A, C) = \{x \in b(A) \colon r(A, x) \leqq C\}$ is dense in X. As $\| Ax \| \leqq C \| x \|$ when $x \in K(A, C)$ and A is closed, $D(A) = X$ and $A \in B(X)$.

REMARK. Operator versions of (2.6), (2.7) have been obtained also by Ditzian, Certain–Kurtz and Chernoff — see [8], but only for generators of C_0 (semi)groups. Here A need not even be densely defined, but we have to put restrictions on the element x.

We know that $r(A) \leqq \|A\|$ for every $A \in B(X)$. In connection with this and (1.5), (2.10), the question arises whether an inequality

$$r(A, x)\|x\| \leqq C\|Ax\| \qquad (C \text{ a constant}) \tag{2.11}$$

holds at least for bounded Hermitian operators. This, however, is not true, even for Hermitian projections on a Hilbert space.

EXAMPLE. Let H be the Hilbert space R^2 with norm $\|(u, v)\| = (u^2 + v^2)^{1/2}$ and let

$$x = (u, (1-u^2)^{1/2}), \quad 0 < u < 1, \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We have $\|x\| = 1$, $r(P, x) = 1$, $\|Px\| = u$. If (2.11) holds, letting $u \to 0$ we come to a contradiction.

Another question: Does an inequality of the form (2.10) hold for dissipative operators too? The answer is "no". Lumer and Phillips in [18, Theorem 2.2] have given an example of a nonzero bounded dissipative quasinilpotent operator acting on a Hilbert space. For such an operator (2.10) is impossible. However, we have the following.

PROPOSITION 1. *Let A be closed, dissipative, and $x \in b(A)$. Then*

$$\|Ax\| \leqq (e/2)\|x\| \liminf_{n\to\infty} (n\|A^n x\|^{1/n}) \tag{2.12}$$

and

$$\|Ax\| \leqq (2/e)\|x\| \limsup_{n\to\infty} (n\|A^n x\|^{1/n}). \tag{2.13}$$

PROOF. The following estimates hold:

$$\frac{2}{e}(1 - o(1))n \leqq C^{+}_{n,1} \leqq \frac{e}{2} n, \qquad n = 2, 3, \ldots$$

(see [22]), which in view of (2.9) imply (2.12).

When

$$\gamma = \limsup_{n\to\infty} (n\|A^n x\|^{1/n}) < \infty,$$

we consider the entire function $f(z) = h(F(A, z^2)x)$, $h \in X'$, $\|h\| = 1$, which is of exponential type

$$\tau = \limsup_{k\to\infty} |f^{(k)}(0)|^{1/k} \leqq \limsup_{n\to\infty} (((2n)!/n!)\|A^n x\|)^{1/2n} = 2(\gamma/e)^{1/2}$$

(by Stirling's formula $((2n)!/n!)^{1/2n} \sim 2(n/e)^{1/2}$ when $n\to\infty$).

As $|f(t)| \leqq \|x\|$ for $t \in R$ (Lemma 2), we can apply Bernstein's inequality (0,1) twice to obtain

$$|f''(0)| = 2|h(Ax)| \leqq \tau^2\|x\| \leqq (4/e)\gamma\|x\|$$

and (2.13) follows.

REMARK. In view of the above it is reasonable to conjecture that

$$\liminf_{n\to\infty}(n\|A^n x\|^{1/n}) = \limsup_{n\to\infty}(n\|A^n x\|^{1/n}) \quad \text{and} \quad C^+_{n,1}/n \to 2/e \quad (n\to\infty).$$

3. A class of operators

In this section we prove some propositions about operators $T\in B(X)$ satisfying for every $x\in X$ the inequality

$$\|Tx\| \leqq C\|x\| r(T,x) \tag{3.1}$$

with some constant $C = C(T)$. We call such operators subjacent.

According to (1.5), every normal equivalent operator has this property. Furuta and Istratescu introduced, some twenty years ago, the class of paranormal (or class (N)) operators, defined by

$$\|Tx\|^2 \leqq \|x\|\,\|T^2x\| \qquad (x\in X) \tag{3.2}$$

(see [13], p. 605 in the same journal and [14]). For them induction implies

$$\|Tx\| \leqq \|x\|^{1-1/n}\|T^n x\|^{1/n}, \qquad n = 2,3,\ldots,$$

hence every paranormal operator satisfies (3.1) with $C = 1$. It is easy to see that every quasihyponormal operator T (i.e. $\|T^*Tx\| \leqq \|T^2x\|$, $x\in X$), hence every hyponormal operator ($\|T^*x\| \leqq \|Tx\|$), is paranormal.

In what follows we assume that $T\in B(X)$ is a subjacent operator.

PROPOSITION 2. *Let $T^kS = SQ$ for some positive integer k and $S, Q \in B(X)$, Q quasinilpotent. Then $TS = 0$.*

PROOF. We have $T^{kn}S = SQ^n$, $\|T^{kn}Sx\|^{1/n} \leqq \|S\|^{1/n}\|Q^n x\|^{1/n}$ ($n = 1,2,\ldots$, $x\in X$). Hence $r(T^k, Sx) = 0$. A result of Apostol [2] says that

$$\{y\in X: r(T^k,y) = 0\} = \{y \in X: r(T,y) = 0\} \qquad (k = 2,3,\ldots)$$

and so $r(T,Sx) = 0$. Now (3.1) implies $TSx = 0$.

PROPOSITION 3. *If $T^k x = Qx$ for some $x \in X$, integer k, and quasinilpotent operator Q commuting with T, then $Tx = 0$.*

PROOF. $T^{kn}x = Q^n x$ $(n = 2, 3, \ldots)$. Hence $r(T^k, x) = 0$ and $r(T, x) = 0$.

Applied to normal equivalent operators, this result together with Corollary 4 in Section 1 provides a further generalization of Fuglede–Putnam's commutation theorem (cf. [11]).

PROPOSITION 4. *If $0 < r(T, x) < \|T\|/C$ for some $x \in X$, then $K = K(T, r(T, x))$ is a non-trivial hyperinvariant subspace for T.*

($C = C(T)$ — see (3.1); recall that $K(T, a) = \{y \in X\colon r(T, y) \leqq a\}$.)

PROOF. $K \neq \{0\}$, because $0 \neq x \in K$. For every $y \in K$ we have

$$\|Ty\| \leqq C\|y\| r(T, y) \leqq Cr(T, x)\|y\|.$$

Suppose now that K is dense in X. Then $\|T\| \leqq Cr(T, x) < \|T\|$ — a contradiction.

Finally, let $S \in B(X)$, $ST = TS$. For every $y \in X$ and $n = 1, 2, \ldots$ we have $\|T^n S y\|^{1/n} \leqq \|S\|^{1/n} \|T^n y\|^{1/n}$, so that $r(T, Sy) \leqq r(T, y)$. It follows that S maps K in K.

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